

# **$B_K$ from improved staggered fermions using SU(3) chiral perturbation theory**

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We present recent progress in our calculation of  $B_K$  with improved staggered fermions using chiral extrapolations based on SU(3) staggered chiral perturbation theory. We have accumulated significantly higher statistics on the coarse, fine, and ultrafine MILC asqtad lattices. This leads to a reduction in statistical error and an improved continuum extrapolation. Our updated result is  $\hat{B}_K = B_K(\text{RG}) = 0.737 \pm 0.003(\text{stat}) \pm 0.046(\text{sys})$ . This is consistent with the result obtained using chiral extrapolations based on SU(2) staggered chiral perturbation theory, although the total error is somewhat larger with the SU(3) analysis.

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## 1. Introduction

Calculations of the kaon mixing parameter  $B_K$  have reached a mature stage with several results available having all errors controlled and small. Our calculation uses improved staggered fermions—HYP-smeared valence on asqtad sea—and at present achieves total error of  $\sim 5\%$  [1, 2]. Our main analysis uses chiral fitting functions derived from SU(2) staggered chiral perturbation theory (SChPT), and is updated in Ref. [2]. As a check on this result, we also carry out an analysis using fit forms from SU(3) SChPT, and in the present report we update the results from this analysis. In particular, we focus on the progress since our publication [1] and last year’s lattice proceedings [3].

In Table 1 we show the current set of ensembles on which we have done the SU(3) analysis, together with the resulting values for  $B_K$ . In the last year, we have accumulated higher statistics on the C2, C5, F1, F2, and U1 ensembles. The most important improvements are those for the F1 and U1 ensembles, since these are used for continuum extrapolation. As described in Ref. [2], the 9-fold increase in statistics on the F1 ensemble significantly impacts the continuum extrapolation in the SU(2) analysis. It forces us to fit to only the three smallest lattice spacings, excluding the coarse ensembles. Our main aim here is to show how the increase in statistics impacts the SU(3) analysis.

$a$ (fm)	$am_\ell/am_s$	geometry	ID	ens $\times$ meas	$B_K$ (N-BB1)	$B_K$ (N-BB2)
0.12	0.03/0.05	$20^3 \times 64$	C1	$564 \times 1$	0.555(12)	0.564(17)
0.12	0.02/0.05	$20^3 \times 64$	C2*	$486 \times 9$	0.538(12)	0.535(17)
0.12	0.01/0.05	$20^3 \times 64$	C3	$671 \times 9$	0.562(6)	0.592(14)
0.12	0.01/0.05	$28^3 \times 64$	C3-2	$275 \times 8$	0.575(6)	0.595(13)
0.12	0.007/0.05	$20^3 \times 64$	C4	$651 \times 10$	0.564(5)	0.598(13)
0.12	0.005/0.05	$24^3 \times 64$	C5*	$509 \times 9$	0.576(5)	0.598(12)
0.09	0.0062/0.031	$28^3 \times 96$	F1*	$995 \times 9$	0.536(3)	0.561(10)
0.09	0.0031/0.031	$40^3 \times 96$	F2*	$850 \times 1$	0.539(7)	0.544(13)
0.06	0.0036/0.018	$48^3 \times 144$	S1	$744 \times 2$	0.535(6)	0.560(11)
0.06	0.0025/0.018	$56^3 \times 144$	S2 <sup>#</sup>	$198 \times 9$	-NA-	-NA-
0.045	0.0028/0.014	$64^3 \times 192$	U1*	$705 \times 1$	0.543(4)	0.554(8)

**Table 1:** MILC asqtad ensembles used in the calculation. Ensembles marked with a \* have improved statistics compared to last year, while those marked with a # are new. Results for  $B_K(\mu = 2 \text{ GeV})$  using both N-BB1 and N-BB2 fits are given. See the text for discussion of these fits.

## 2. Fitting and Results

In our numerical study, our lattice kaon is composed of valence (anti-)quarks with masses  $m_x$  and  $m_y$ . On each MILC asqtad ensemble, we use 10 valence masses:

$$am_x, am_y = am_s \times n/10 \quad \text{with} \quad n = 1, 2, 3, \dots, 10 \quad (2.1)$$

where  $am_s$  is the nominal strange sea quark mass of the given lattice ensemble. Hence, we have 55 combinations of the lattice kaon: 10 degenerate ( $m_x = m_y$ ) and 45 non-degenerate ( $m_x \neq m_y$ ).

An important difference between the SU(2) and SU(3) analyses is the number of the mass combinations that are included. In the SU(2) analysis, we use only those in which the valence  $d$  quark is much lighter than the valence  $s$  quark. Specifically, we use the lightest 4 or 5 values of  $m_x$  and the heaviest 3 values of  $m_y$ . In the SU(3) analysis, by contrast, we use all 55 combinations. While this makes better use of our data, it does so at considerable cost. First, quite a few of our mass combinations are in the regime where next-to-leading order (NLO) SU(3) ChPT is beginning to break down. Second, the fit forms in SU(3) SChPT contain very many fit parameters [4] and have to be simplified by hand in order to be practical. In the end, as explained in Ref. [1], we came up with two different schemes for fitting, both using Bayesian constraints on parameters which arise due to discretization errors, but doing so in somewhat different ways. We focus here on the results for these two schemes, which we call “N-BB1” and “N-BB2”. Both schemes are based on NLO SChPT with the addition of a single analytic next-to-next-to-leading order term.

In the N-BB1 scheme, we fit the data in two stages. First, we fit the 10 degenerate mass combinations to the functional form of Eq. (62) of Ref. [1] using the Bayesian method. We constrain the coefficient of the term arising from discretization and matching errors assuming that discretization errors dominate, so that  $c_4 \propto (a\Lambda_{\text{QCD}})^2$ . Second, we fit all 55 combinations to the fitting functional form of Eq. (63) of Ref. [1] using the Bayesian method. In this second stage, we make similar assumptions concerning the size of terms arising from discretization errors, and also input the results from the degenerate fit (for a subset of the total set of parameters) as Bayesian constraints.

The N-BB2 scheme differs only in that we assume that matching errors dominate in the low-energy coefficients arising from discretization and matching errors, so that, for example,  $c_4 \propto \alpha_s^2$ , (where we evaluate  $\alpha_s$  at the scale of  $\mu = 1/a$  in the  $\overline{\text{MS}}$  scheme).

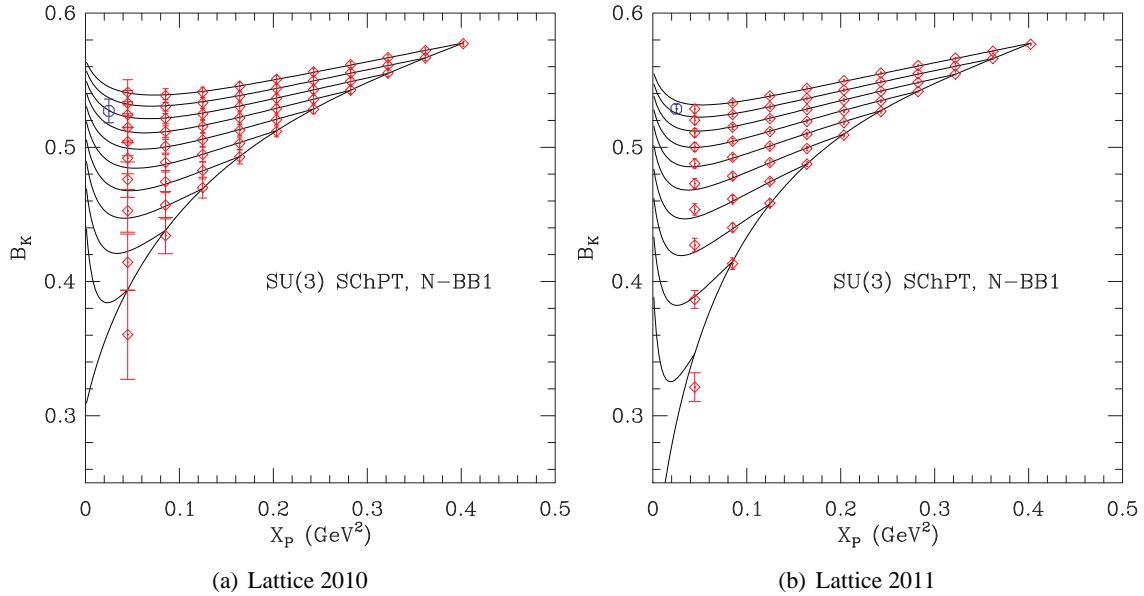
In Fig. 1, we show how the 9-fold increase in statistics impacts the N-BB1 fits on the F1 ensemble. As expected, errors have been reduced by a factor of  $\sim 3$ , and this holds also for the value for  $B_K$  that results after extrapolation to physical valence-quark masses and removal of taste-breaking lattice artifacts. This final value (shown as a blue octagon in the figure) has shifted up by about  $0.2\sigma$ .

The changes in the N-BB2 fits (not shown) are different: the final value is shifted up by  $1.8\sigma$  while the error is reduced only by a factor of  $\sim 1.2$ .

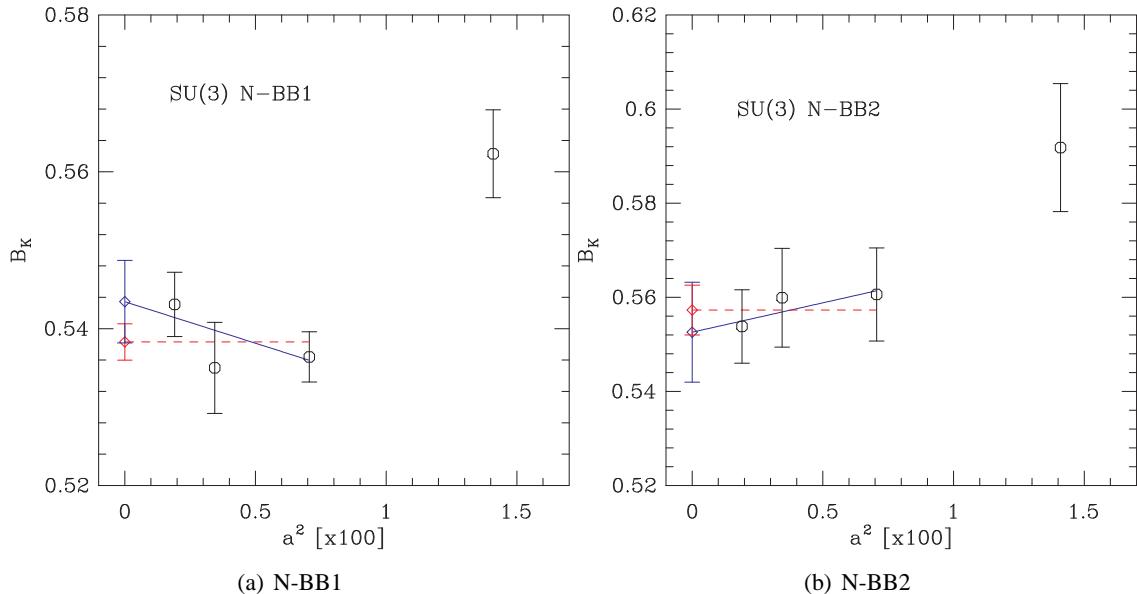
### 3. Continuum extrapolation

In Fig 2, we present our updated results for the continuum extrapolation using both N-BB1 and N-BB2 fits. In the case of the N-BB1 fit, it is not possible to obtain a good fit to all four values of  $a$  using a simple fitting functional form, e.g.  $c_1 + c_2 a^2 + c_3 a^4$ , with physically reasonable values for the coefficients. This is the same issue that arises in the SU(2) fits, and is discussed in that case in more detail in the companion proceedings [2]. We proceed by fitting only to the smallest three values of  $a$ , using either constant or linear fits.

For the N-BB2 fit we do obtain reasonable fits to all four values of  $a$ , but, in order to compare with the N-BB1 fits, we also use only the smallest three values of  $a$ .



**Figure 1:**  $B_K(1/a)$  versus  $X_P$  (squared mass of pion composed of valence  $x$  and  $\bar{x}$ ) for the F1 ensemble, using N-BB1 fits. The left panel uses 1 measurement/configuration, while the right panel uses 9 measurements/configuration. Red diamonds show the data, while the blue octagon shows the result obtained after extrapolation to physical quark masses with all taste-breaking lattice artifacts removed.



**Figure 2:**  $B_K(\text{NDR}, \mu = 2 \text{ GeV})$  as a function of  $a^2$  (in  $\text{fm} \times 100$ ) for the N-BB1 fit (left) and the N-BB2 fit (right), showing constant and linear extrapolations to the smallest three values of  $a$ .

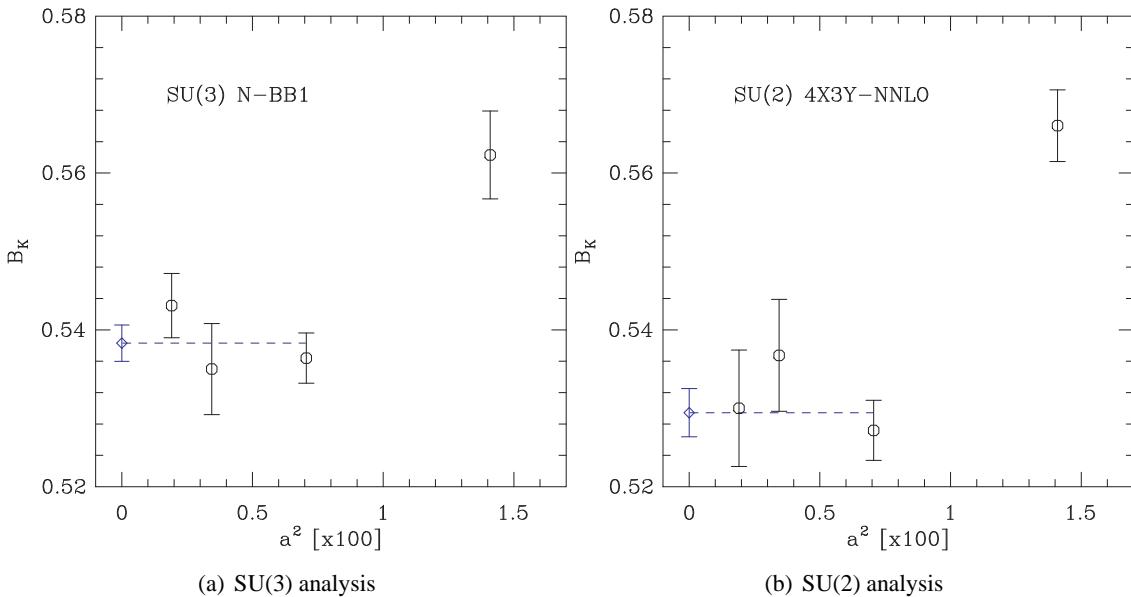
In Table 2, we summarize the results of these continuum extrapolations. We use the constant fit to the N-BB1 data for our central value and the difference between this and the result from the corresponding fit to the N-BB2 data as an estimate of the fitting systematic. Note that the difference between N-BB1 and N-BB2 fits is much larger than the difference between constant and

fit type	constant fit	linear fit
N-BB1	0.5383(23)	0.5344(53)
N-BB2	0.5573(53)	0.5526(106)

**Table 2:** Results for  $B_K(\mu = 2 \text{ GeV})$  to data from N-BB1 and N-BB2 chiral fits using both constant and linear continuum extrapolations (in  $a^2$ ).

linear extrapolations. This reflects the uncertainty in SU(3) fitting caused by the large number of parameters related to discretization and matching errors.

In Fig. 3 we compare the results from the N-BB1 fits to those using our preferred SU(2) fitting approach. There is reasonable consistency point by point.



**Figure 3:**  $B_K(\text{NDR}, \mu = 2 \text{ GeV})$  as a function of  $a^2$  (in  $\text{fm} \times 100$ ) from the SU(3) analysis (N-BB1 fit) in the left panel and from the SU(2) analysis in the right panel. Constant fits to the three smallest values of  $a = 0$  are shown.

#### 4. Error Budget and Conclusions

In Table 3, we list various sources of error in the SU(3)-based calculation of  $B_K$ . Many of the smaller errors are unchanged from Ref. [1], and we refer to that paper for an explanation of how they are estimated.

The major changes to the errors are as follows. The statistical error has been reduced from 1.4% in Ref. [1] to 0.4%, due to the use of more measurements. The ‘‘matching’’ error—due to our use of one-loop matching between lattice and continuum operators—has been reduced from 5.5% in Ref. [1] to 4.4%. This is simply due to the addition of the smallest lattice spacing, for our estimate of the percentage error is  $\alpha_s(\mu = 1/a_{\min})^2$ . The discretization error has also been reduced (from 2.2%), since our smallest value of  $a$  is closer to the continuum. Finally, the ‘‘fitting (2)’’

cause	error (%)	memo	status
statistics	0.43	N-BB1 fit	update
matching factor	4.4	$\Delta B_K^{(2)}$ (U1)	update
discretization	0.95	diff. of const and linear extrap	update
fitting (1)	0.36	diff. of N-BB1 and N-B1 (C3)	[1]
fitting (2)	3.5	diff. of N-BB1 and N-BB2 at $a = 0$	update
$am_l$ extrap	1.0	diff. of (C3) and linear extrap	[1]
$am_s$ extrap	0.5	constant vs. linear extrap	[1]
finite volume	2.3	diff. of $20^3$ (C3) and $28^3$ (C3-2)	[1]
$r_1$	0.12	error propagation from $r_1$	[1]

**Table 3:** Error budget for  $B_K$  obtained using SU(3) SChPT fitting.

error—that due to the uncertainty in the size of SChPT coefficients introduced by discretization and matching errors—has changed from 5.3% to 3.5%. Nevertheless, this error remains large, and along with the matching error, dominates the total error. This large fitting systematic is a reflection of the difficulties in using SU(3) SChPT and is reason why we think that the SU(2) analysis is more reliable.

Our present value from the SU(3) analysis is

$$\begin{aligned} B_K(\text{NDR}, \mu = 2 \text{ GeV}) &= 0.5383 \pm 0.0023 \pm 0.0337, & \text{SU(3) fit} \\ \hat{B}_K = B_K(\text{RG1}) &= 0.7371 \pm 0.0032 \pm 0.0461, & \text{SU(3) fit} \end{aligned} \quad (4.1)$$

where the first error is statistical and the second systematic. The total error is 6.3%. This should be compared to our updated SU(2) result [2]

$$\hat{B}_K = B_K(\text{RG1}) = 0.725 \pm 0.004(\text{stat}) \pm 0.038(\text{sys}) \quad \text{SU(2) fit.} \quad (4.2)$$

We see that, although the SU(3) result has a smaller statistical error, it has a significantly larger systematic error. The most important observation, however, is that the two results are consistent.

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